

BLOCKS OF DEFECT OF p -SOLVABLE GROUPS

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ABSTRACT. Let p be a prime such that $p \geq 5$. Let G be a finite p -solvable group and let p^a be the largest power of p dividing $\chi(1)$ for an irreducible character χ of G , we show that $|G : \mathbf{F}(G)|_p \leq p^{5.5a}$. Let G be a finite p -solvable group with trivial maximal normal solvable subgroup and we denote $|G|_p = p^n$, then G contains a block of defect less than or equal to $\lfloor \frac{2n}{3} \rfloor$.

1. INTRODUCTION

Let G be a finite group. Let p be a prime and $|G|_p = p^n$. An irreducible ordinary character of G is called p -defect zero if and only if its degree is divisible by p^n . It is an interesting problem to give necessary and sufficient conditions for the existence of p -blocks of defect zero. If a finite group G has a character of p -defect zero, then $O_p(G) = 1$ [4, Corollary 6.9]. Unfortunately, the converse is not true.

Although the block of defect zero may not exist in general, one could try to find the smallest defect $d(B)$ of a block B of G . In [3, Theorem A], Espuelas and Navarro raised the following question. If G is a finite group with $O_p(G) = 1$ for some prime $p \geq 5$, and denote $|G|_p = p^n$, does G contain a block of defect less than or equal to $\lfloor \frac{n}{2} \rfloor$?

In [23], the author provided a partial answer to the previous question under the condition where G is solvable. In this paper, we prove a related result about p -solvable groups.

Theorem A. *Let p be a prime such that $p \geq 5$. Let G be a finite p -solvable group such that $O_\infty(G) = 1$, and we denote $|G|_p = p^n$. Then G contains a p -block B such that $d(B) \leq \lfloor \frac{2n}{3} \rfloor$.*

Let p^a denote the largest power of p dividing $\chi(1)$ for an irreducible character χ of G . Moretó and Wolf [18, Theorem A] proved that for G solvable, there exists a product $\theta = \chi_1(1) \cdots \chi_t(1)$ of distinct irreducible characters χ_i such that $|G : \mathbf{F}(G)|$ divides $\theta(1)$ and $t \leq 19$. This implies that $|G : \mathbf{F}(G)|_p \leq p^{19a}$. In this paper, we show the following result for p -solvable groups.

Theorem B. *Let G be a p -solvable group where p is a prime and $p \geq 5$. Suppose that p^{a+1} does not divide $\chi(1)$ for all $\chi \in \text{Irr}(G)$, then $|G : \mathbf{F}(G)|_p \leq p^{5.5a}$.*

We first fix some notation:

- (1) We use $\mathbf{F}(G)$ to denote the Fitting subgroup of G . Let $\mathbf{F}_0(G) \leq \mathbf{F}_1(G) \leq \mathbf{F}_2(G) \leq \cdots \leq \mathbf{F}_n(G) = G$ denote the ascending Fitting series, i.e. $\mathbf{F}_0(G) = 1$, $\mathbf{F}_1(G) = \mathbf{F}(G)$ and $\mathbf{F}_{i+1}(G)/\mathbf{F}_i(G) = \mathbf{F}(G/\mathbf{F}_i(G))$. $\mathbf{F}_i(G)$ is the i th ascending Fitting subgroup of G .
- (2) We use $F^*(G)$ to denote the generalized Fitting subgroup of G .
- (3) We use $O_\infty(G) = 1$ to denote the maximal normal solvable subgroup of G .
- (4) Let G be a finite group, we denote $cd(G) = \{\chi(1) \mid \chi \in \text{Irr}(G)\}$.

2. BLOCKS OF SMALL DEFECT

Let G be a finite group. Let p be a prime and $|G|_p = p^n$. An irreducible ordinary character of G is called p -defect 0 if and only if its degree is divisible by p^n . By [4, Theorem 4.18], G has a character of p -defect 0 if and only if G has a p -block of defect 0. An important question in the modular representation theory of finite groups is to find the group-theoretic conditions for the existence of characters of p -defect 0 in a finite group.

It is an interesting problem to give necessary and sufficient conditions for the existence of p -blocks of defect zero. If a finite group G has a character of p -defect 0, then $O_p(G) = 1$ [4, Corollary 6.9]. Unfortunately, the converse is not true. Zhang [24] and Fukushima [9, 10] provided various sufficient conditions where a finite group G has a block of defect zero.

Although the block of defect zero may not exist in general, one could try to find the smallest defect $d(B)$ of a block B of G . One of the results along this line is given by [3, Theorem A]. In [3], Espuelas and Navarro bounded the smallest defect $d(B)$ of a block B of G using the p -part of G . Using an orbit theorem [2, Theorem 3.1] of solvable linear groups of odd order, they showed the following result. Let G be a (solvable) group of odd order such that $O_p(G) = 1$ and $|G|_p = p^n$, then G contains a p -block B such that $d(B) \leq \lfloor \frac{n}{2} \rfloor$. The bound is best possible, as shown by an example in [3].

It is not true in general that there exists a block B with $d(B) \leq \lfloor \frac{n}{2} \rfloor$, as $G = A_7(p = 2)$ shows us. However, the counterexamples were only found for $p = 2$ and $p = 3$. By work of Michler and Willems [15, 21] every simple group except possibly the alternating group has a block of defect zero for $p \geq 5$. The alternating group case was settled by Granville and Ono in [8] using number theory.

Based on these, the following question raised by Espuelas and Navarro [3] seems to be natural. If G is a finite group with $O_p(G) = 1$ for some prime $p \geq 5$, and denote $|G|_p = p^n$, does G contain a block of defect less than $\lfloor \frac{n}{2} \rfloor$?

In this section, we study this question and show that for a p -solvable group G where $O_\infty(G) = 1$ and $p \geq 5$, G contains a block of defect less than or equal to $\lfloor \frac{2n}{3} \rfloor$.

We need the following results about simple groups.

Lemma 2.1. *Let A act faithfully and coprimely on a nonabelian simple group S . Then A has at least 2 regular orbits on $\text{Irr}(S)$.*

Proof. This is [19, Proposition 2.6]. □

Theorem 2.2. *Let G be a non-abelian finite simple group. Then $|cd(G)| \geq 4$,*

Proof. This is Theorem C of [14]. □

Proposition 2.3. *Let S be a nonabelian simple group, and let p be a prime such that $p \geq 3$ and p does not divide $|S|$. Suppose $V = S_1 \times \cdots \times S_n$ where $S_i \cong S$. Assume G is a p -solvable group that acts faithfully on V via automorphisms, and assume the action of G transitively permutes the S_i 's.*

- (1) *Assume $p \geq 5$, then there exists $N \triangleleft G$, $N \subseteq \mathbf{F}_2(G)$ and there exist four G -orbits with representatives v_1, v_2, v_3 and $v_4 \in \text{Irr}(V)$ such that for any $P \in \text{Syl}_p(G)$, we have $\mathbf{C}_P(v_i) \subseteq N$ for $1 \leq i \leq 4$. Moreover, the Sylow p -subgroup of $\mathbf{N}\mathbf{F}(G)/\mathbf{F}(G)$ is abelian.*
- (2) *Assume $p = 3$, then there exists $N \triangleleft G$, $N \subseteq \mathbf{F}_3(G)$ and there exist four G -orbits with representatives v_1, v_2, v_3 and $v_4 \in \text{Irr}(V)$ such that for any $P \in \text{Syl}_p(G)$, we have*

$\mathbf{C}_P(v_i) \subseteq N$ for $1 \leq i \leq 4$. Moreover, the Sylow p -subgroup of $N\mathbf{F}_2(G)/\mathbf{F}_2(G)$ is abelian.

Proof. Clearly G is embedded in $\text{Out}(V) = \text{Out}(S) \wr S_n$. Set $K := G \cap \text{Out}(S)^n$, and thus G/K is a permutation group on n letters.

Assume $n = 1$, then the result follows from Theorem 2.2 and the fact that $\text{Out}(S)$ has a normal series of the form $A \triangleleft B \triangleleft C$ where A is abelian, B/A is cyclic and $C/B \cong 1, S_2$ or S_3 .

Assume $n > 1$, and we first assume that G/K is primitive. Since S has four characters of different degrees by Theorem 2.2, we may denote them to be θ, λ, χ and ψ and we may assume that $\theta(1) > \lambda(1) > \chi(1) > \psi(1)$.

Assume $p \nmid |G/K|$, we may choose $v_1 = \theta^n, v_2 = \lambda^n, v_3 = \chi^n$ and $v_4 = \psi^n$. It is Clear that $\theta(1)^n > \lambda(1)^n > \chi(1)^n > \psi(1)^n$, and $\text{Syl}_p(\mathbf{C}_G(v_i)) \subseteq \text{Out}(S)^n$ for $1 \leq i \leq 4$. Since $\text{Out}(S)$ has a normal series of the form $A \triangleleft B \triangleleft C$ where A is abelian, B/A is cyclic and $C/B \cong 1, S_2$ or S_3 , the result is clear.

Assume $p \mid |G/K|$ and $n \geq 5$, then since G/K is p -solvable, we know that $\text{Alt}(n) \not\leq G/K$. By [1, Lemma 1](b), we can find a partition $\Omega = \Omega_1 \cup \Omega_2 \cup \Omega_3 \cup \Omega_4$ such that $\text{Stab}_{G/K}(\Omega_1) \cap \text{Stab}_{G/K}(\Omega_2) \cap \text{Stab}_{G/K}(\Omega_3) \cap \text{Stab}_{G/K}(\Omega_4)$ is a 2-group and t_1, t_2, t_3 and t_4 are not all the same. We denote $t_i = |\Omega_i|$, $1 \leq i \leq 4$. By re-indexing, we may assume that $t_1 \geq t_2 \geq t_3 \geq t_4$.

Since we know that not all the t_i s are the same, WLOG, we must have one of the followings,

- (1) $t_1 > t_2 \geq t_3 \geq t_4$. In this case, we construct four irreducible characters
 - (a) $\alpha = \prod_{i=1}^n \alpha_i$, where $\alpha_i = \theta_i$ if $i \in \Omega_1$, $\alpha_i = \lambda_i$ if $i \in \Omega_2$, $\alpha_i = \chi_i$ if $i \in \Omega_3$, $\alpha_i = \psi_i$ if $i \in \Omega_4$.
 - (b) $\beta = \prod_{i=1}^n \beta_i$, where $\beta_i = \lambda_i$ if $i \in \Omega_1$, $\beta_i = \theta_i$ if $i \in \Omega_2$, $\beta_i = \chi_i$ if $i \in \Omega_3$, $\beta_i = \psi_i$ if $i \in \Omega_4$.
 - (c) $\gamma = \prod_{i=1}^n \gamma_i$, where $\gamma_i = \chi_i$ if $i \in \Omega_1$, $\gamma_i = \theta_i$ if $i \in \Omega_2$, $\gamma_i = \lambda_i$ if $i \in \Omega_3$, $\gamma_i = \psi_i$ if $i \in \Omega_4$.
 - (d) $\delta = \prod_{i=1}^n \delta_i$, where $\delta_i = \psi_i$ if $i \in \Omega_1$, $\delta_i = \theta_i$ if $i \in \Omega_2$, $\delta_i = \lambda_i$ if $i \in \Omega_3$, $\delta_i = \chi_i$ if $i \in \Omega_4$.

Those four characters have different degrees since

$$\alpha(1) = \theta(1)^{t_1} \lambda(1)^{t_2} \chi(1)^{t_3} \psi(1)^{t_4} > \beta(1) = \lambda(1)^{t_1} \theta(1)^{t_2} \chi(1)^{t_3} \psi(1)^{t_4} > \gamma(1) = \chi(1)^{t_1} \theta(1)^{t_2} \lambda(1)^{t_3} \psi(1)^{t_4} > \delta(1) = \psi(1)^{t_1} \theta(1)^{t_2} \lambda(1)^{t_3} \chi(1)^{t_4}.$$

- (2) $t_1 = t_2 > t_3 \geq t_4$. In this case, we construct four irreducible characters
 - (a) $\alpha = \prod_{i=1}^n \alpha_i$, where $\alpha_i = \theta_i$ if $i \in \Omega_1$, $\alpha_i = \lambda_i$ if $i \in \Omega_2$, $\alpha_i = \chi_i$ if $i \in \Omega_3$, $\alpha_i = \psi_i$ if $i \in \Omega_4$.
 - (b) $\beta = \prod_{i=1}^n \beta_i$, where $\beta_i = \theta_i$ if $i \in \Omega_1$, $\beta_i = \chi_i$ if $i \in \Omega_2$, $\beta_i = \lambda_i$ if $i \in \Omega_3$, $\beta_i = \psi_i$ if $i \in \Omega_4$.
 - (c) $\gamma = \prod_{i=1}^n \gamma_i$, where $\gamma_i = \theta_i$ if $i \in \Omega_1$, $\gamma_i = \psi_i$ if $i \in \Omega_2$, $\gamma_i = \lambda_i$ if $i \in \Omega_3$, $\gamma_i = \chi_i$ if $i \in \Omega_4$.
 - (d) $\delta = \prod_{i=1}^n \delta_i$, where $\delta_i = \chi_i$ if $i \in \Omega_1$, $\delta_i = \psi_i$ if $i \in \Omega_2$, $\delta_i = \lambda_i$ if $i \in \Omega_3$, $\delta_i = \theta_i$ if $i \in \Omega_4$.

Those four characters have different degrees since

$$\alpha(1) = \theta(1)^{t_1} \lambda(1)^{t_2} \chi(1)^{t_3} \psi(1)^{t_4} > \beta(1) = \theta(1)^{t_1} \chi(1)^{t_2} \lambda(1)^{t_3} \psi(1)^{t_4} > \gamma(1) = \theta(1)^{t_1} \psi(1)^{t_2} \lambda(1)^{t_3} \chi(1)^{t_4} > \delta(1) = \chi(1)^{t_1} \psi(1)^{t_2} \lambda(1)^{t_3} \theta(1)^{t_4}.$$

- (3) $t_1 = t_2 = t_3 > t_4$. In this case, we construct four irreducible characters

- (a) $\alpha = \prod_{i=1}^n \alpha_i$, where $\alpha_i = \theta_i$ if $i \in \Omega_1$, $\alpha_i = \lambda_i$ if $i \in \Omega_2$, $\alpha_i = \chi_i$ if $i \in \Omega_3$, $\alpha_i = \psi_i$ if $i \in \Omega_4$.
- (b) $\beta = \prod_{i=1}^n \beta_i$, where $\beta_i = \theta_i$ if $i \in \Omega_1$, $\beta_i = \lambda_i$ if $i \in \Omega_2$, $\beta_i = \psi_i$ if $i \in \Omega_3$, $\beta_i = \chi_i$ if $i \in \Omega_4$.
- (c) $\gamma = \prod_{i=1}^n \gamma_i$, where $\gamma_i = \theta_i$ if $i \in \Omega_1$, $\gamma_i = \chi_i$ if $i \in \Omega_2$, $\gamma_i = \psi_i$ if $i \in \Omega_3$, $\gamma_i = \lambda_i$ if $i \in \Omega_4$.
- (d) $\delta = \prod_{i=1}^n \delta_i$, where $\delta_i = \lambda_i$ if $i \in \Omega_1$, $\delta_i = \chi_i$ if $i \in \Omega_2$, $\delta_i = \psi_i$ if $i \in \Omega_3$, $\delta_i = \theta_i$ if $i \in \Omega_4$.

Those four characters have different degrees since

$$\begin{aligned} \alpha(1) &= \theta(1)^{t_1} \lambda(1)^{t_2} \chi(1)^{t_3} \psi(1)^{t_4} > \beta(1) = \theta(1)^{t_1} \lambda(1)^{t_2} \psi(1)^{t_3} \chi(1)^{t_4} > \\ \gamma(1) &= \theta(1)^{t_1} \chi(1)^{t_2} \psi(1)^{t_3} \lambda(1)^{t_4} > \delta(1) = \lambda(1)^{t_1} \chi(1)^{t_2} \psi(1)^{t_3} \theta(1)^{t_4}. \end{aligned}$$

Assume $p = 3$, $p \mid |G/K|$ and $n = 4$. In this case, we construct four irreducible characters $\alpha = \theta\lambda\chi\psi$, $\beta = \theta\lambda\psi\psi$, $\gamma = \theta\chi\psi\psi$ and $\delta = \lambda\chi\psi\psi$.

Those four characters have different degrees since $\alpha(1) = \theta(1)\lambda(1)\chi(1)\psi(1) > \beta(1) = \theta(1)\lambda(1)\psi^2(1) > \gamma(1) = \theta(1)\chi(1)\psi^2(1) > \delta(1) = \lambda(1)\chi(1)\psi^2(1)$.

It is clear that $\mathbf{C}_{G/K}(\alpha) = \mathbf{C}_{G/K}(\beta) = \mathbf{C}_{G/K}(\gamma) = \mathbf{C}_{G/K}(\delta) = 1$.

Assume $p = 3$, $p \mid |G/K|$ and $n = 3$. In this case, we construct four irreducible characters, $\alpha = \theta\lambda\chi$, $\beta = \theta\lambda\psi$, $\gamma = \theta\chi\psi$ and $\delta = \lambda\chi\psi$.

Those four characters have different degrees since $\alpha(1) = \theta(1)\lambda(1)\chi(1) > \beta(1) = \theta(1)\lambda(1)\psi(1) > \gamma(1) = \theta(1)\chi(1)\psi(1) > \delta(1) = \lambda(1)\chi(1)\psi(1)$.

It is clear that $\mathbf{C}_{G/K}(\alpha), \mathbf{C}_{G/K}(\beta), \mathbf{C}_{G/K}(\gamma), \mathbf{C}_{G/K}(\delta)$ is a 2-group.

Thus, we may always find four irreducible characters $v_1 = \alpha$, $v_2 = \beta$, $v_3 = \gamma$ and $v_4 = \delta$ in $\text{Irr}(V)$ of different degrees such that for any $P \in \text{Syl}_p(G)$, we have $\mathbf{C}_G(v_i) \subseteq \text{Out}(S)^n$ for $1 \leq i \leq 4$. Since $\text{Out}(S)$ has a normal series of the form $A \triangleleft B \triangleleft C$ where A is abelian, B/A is cyclic and $C/B \cong 1, S_2$ or S_3 , the result is clear.

Assume $n > 1$, and we assume that G/K is not primitive. The argument follows by induction. \square

Proposition 2.4. *Let G be a finite p -solvable group where $O_\infty(G) = 1$. Then $F^*(G) = E_1 \times \cdots \times E_m$ is a product of m finite non-abelian simple groups E_i , $1 \leq i \leq m$ permuted by G . Let $L = \bigcap_i \mathbf{N}_G(E_i)$. Clearly $L/F^*(G) \leq \text{Out}E_1 \times \cdots \times \text{Out}E_m$, and we denote $\bar{G} = G/F^*(G)$.*

- (1) *Assume $p \geq 5$, then there exists $N \triangleleft \bar{G}$ where $N \subseteq \mathbf{F}_2(\bar{G})$, and there exists $v \in \text{Irr}(F^*(G))$ such that for any $P \in \text{Syl}_p(\bar{G})$, we have $\mathbf{C}_P(v) \subseteq N$. Moreover, the Sylow p -subgroup of $N\mathbf{F}(\bar{G})/\mathbf{F}(\bar{G})$ is abelian.*
- (2) *Assume $p = 3$, then there exists $N \triangleleft \bar{G}$ where $N \subseteq \mathbf{F}_3(\bar{G})$, and there exists $v \in \text{Irr}(F^*(G))$ such that for any $P \in \text{Syl}_p(\bar{G})$, we have $\mathbf{C}_P(v) \subseteq N$. Moreover, the Sylow p -subgroup of $N\mathbf{F}_2(\bar{G})/\mathbf{F}_2(\bar{G})$ is abelian.*

Proof. Consider $K = G/F^*(G)$, K acts faithfully on $F^*(G)$. Next, we group the simple groups in the direct product of $F^*(G)$ where K acts transitively. We denote $F^*(G) = L_1 \times \cdots \times L_s$, $L_j = E_{j1} \times E_{j2} \times \cdots \times E_{jm_j}$, $1 \leq j \leq s$ where K transitively permutes the simple groups inside the direct product of L_j . Clearly $E_{j1} \cong E_{j2} \cdots \cong E_{jm_j}$.

We see that K can be embedded as a subgroup of $K/\mathbf{C}_K(L_1) \times \cdots \times K/\mathbf{C}_K(L_s)$ and we denote $K_i = K/\mathbf{C}_K(L_i)$, $C_i = \mathbf{C}_K(L_i)$.

If $p \geq 5$, by applying Proposition 2.3(1) to the action of K_i on L_i , there exists $v_i \in \text{Irr}(L_i)$, and $N_i \triangleleft K_i$ such that for any $P_i \in \text{Syl}_p(K_i)$, $\mathbf{C}_{P_i}(v_i) \subseteq N_i \subseteq \mathbf{F}_2(K_i)$. Also the Sylow p -subgroup of $N_i \mathbf{F}(K_i)/\mathbf{F}(K_i)$ is abelian.

Let $v = \sum v_i$ and $N = K \cap (\prod N_i)$. Let $P \in \text{Syl}_p(G)$ and $P_i = PC_i/C_i$. $\mathbf{C}_P(v) \subseteq \prod \mathbf{C}_{P_i}(v_i) \subseteq \prod N_i$. Clearly $N\mathbf{F}(K)/\mathbf{F}(K) \subseteq \prod N_i \mathbf{F}(K_i)/\mathbf{F}(K_i)$ and the result follows.

If $p = 3$, by applying Proposition 2.3(2) to the action of K_i on L_i , there exists $v_i \in \text{Irr}(L_i)$, and $N_i \triangleleft K_i$ such that for any $P_i \in \text{Syl}_p(K_i)$, $\mathbf{C}_{P_i}(v_i) \subseteq N_i \subseteq \mathbf{F}_3(K_i)$. Also the Sylow p -subgroup of $N_i \mathbf{F}_2(K_i)/\mathbf{F}_2(K_i)$ is abelian.

Let $v = \sum v_i$ and $N = K \cap (\prod N_i)$. Let $P \in \text{Syl}_p(G)$ and $P_i = PC_i/C_i$. $\mathbf{C}_P(v) \subseteq \prod \mathbf{C}_{P_i}(v_i) \subseteq \prod N_i$. Clearly $N\mathbf{F}_3(K)/\mathbf{F}_3(K) \subseteq \prod N_i \mathbf{F}_3(K_i)/\mathbf{F}_3(K_i)$ and the result follows. \square

Theorem A. *Let p be a prime such that $p \geq 5$. Let G be a finite p -solvable group such that $O_\infty(G) = 1$, and we denote $|G|_p = p^n$. Then G contains a p -block B such that $d(B) \leq \lfloor \frac{2n}{3} \rfloor$.*

Proof. Since $O_\infty(G) = 1$, $F^*(G) = L_1 \times \cdots \times L_m$ is the product of m finite non-abelian simple groups permuted by $G/F^*(G)$. We denote $\bar{G} = G/F^*(G)$ and let $J = \bigcap_i \mathbf{N}_G(L_i)$. Clearly $J/F^*(G) \leq \text{Out} L_1 \times \cdots \times \text{Out} L_m$. By Proposition 2.4(1), there exists $N \triangleleft \bar{G}$ where $N \subseteq \mathbf{F}_2(\bar{G})$ and one \bar{G} -orbit with representative $\lambda \in \text{Irr}(F^*(G))$ such that for any $P \in \text{Syl}_p(\bar{G})$, we have $\mathbf{C}_P(v) \subseteq N$. Furthermore, the Sylow p -subgroup of $N\mathbf{F}(\bar{G})/\mathbf{F}(\bar{G})$ is abelian.

Let $p^n = |\bar{G}|_p$, $p^{n_1} = |N \cap \mathbf{F}(\bar{G})|_p$, $p^{n_2} = |N\mathbf{F}(\bar{G}) : \mathbf{F}(\bar{G})|_p$ and $p^{n_3} = |\bar{G} : N|_p$. Clearly $n = n_1 + n_2 + n_3$.

Take $\chi \in \text{Irr}(G)$ lying over λ and let B be the p -block of G containing χ . As $F^*(G)$ is a p' -group, [4, Lemma V.2.3] shows that every irreducible character ψ in B has λ as an irreducible constituent. Now $\psi(1)_p \geq |\bar{G} : \mathbf{C}_{\bar{G}}(\lambda)|_p \geq p^{n_3}$.

Let $P/\mathbf{F}(\bar{G})$ be a Sylow p -subgroup of $N\mathbf{F}(\bar{G})/\mathbf{F}(\bar{G})$. Where $Y = O_{p'}(\mathbf{F}(\bar{G}))$, observe that $W = \text{Irr}(Y/\Phi(Y))$ is a faithful and completely reducible $P/\mathbf{F}(\bar{G})$ -module. By Gow's regular orbit theorem [7, 2.6], we have $\mu \in W$ such that $\mathbf{C}_P(\mu) = \mathbf{F}(\bar{G})$. We may view μ as a character of the preimage X of Y in \bar{G} . Observe that X is a p' -group. Take $\chi \in \text{Irr}(\bar{G})$ lying over μ , and let B be the p -block of G containing χ . [4, Lemma V.2.3] shows that every irreducible character ψ in B has λ as an irreducible constituent. Now χ lies over an irreducible character ψ of P lying over μ . Clearly, $\psi(1)_p \geq |N\mathbf{F}(\bar{G})/\mathbf{F}(\bar{G})|_p \geq p^{n_2}$. As P is normal in G , we have $\chi(1)_p \geq \psi(1)_p$.

Let P_1 be the Sylow p -subgroup of $\mathbf{F}(\bar{G}) \cap N$. By Lemma 2.1, we may find $\nu \in \text{Irr}(F^*(G))$ such that $\mathbf{C}_{P_1}(\nu) = 1$. Thus by a similar argument as before, we may find a p -block B of G such that for every irreducible character ψ in B , $\psi(1)_p \geq |N \cap \mathbf{F}(\bar{G})|_p = p^{n_1}$.

We know there is a block B such that for every irreducible character α in B , $\alpha(1)_p \geq \max(p^{n_3}, p^{n_2}, p^{n_1})$. Since $n_1 + n_2 + n_3 = n$, it is not hard to see that $\alpha(1)_p \geq p^{\lceil \frac{n}{3} \rceil}$ and thus $d(B) \leq \lfloor \frac{2n}{3} \rfloor$. \square

3. p PART OF $|G : \mathbf{F}(G)|$, CHARACTER DEGREES AND CONJUGACY CLASS SIZES

If P is a Sylow p -subgroup of a finite group G it is reasonable to expect that the degrees of irreducible characters of G somehow restrict those of P . Let p^a denote the largest power of p dividing $\chi(1)$ for an irreducible character χ of G and $b(P)$ denote the largest degree of an irreducible character of P . Conjecture 4 of Moretó [17] suggested $\log b(P)$ is bounded as a function of a . Moretó and Wolf [18] have proven this for G solvable and even something a

bit stronger, namely the logarithm to the base of p of the p -part of $|G : \mathbf{F}(G)|$ is bounded in terms of a . In fact, they showed that $|G : \mathbf{F}(G)|_p \leq p^{19a}$. Moretó and Wolf [18] also proved that $|G : \mathbf{F}(G)|_p \leq p^{2a}$ for odd order groups, this can also be deduced from [3]. This bound is best possible, as shown by an example in [3].

In this paper, we show that for p -solvable groups where $p \geq 5$, $|G : \mathbf{F}(G)|_p \leq p^{5.5a}$.

Theorem 3.1. *Let G be a p -solvable group where p is a prime and $p \geq 5$. Suppose that p^{a+1} does not divide $\chi(1)$ for all $\chi \in \text{Irr}(G)$ and let $P \in \text{Syl}_p(G)$, then $|G : \mathbf{F}(G)|_p \leq p^{5.5a}$, $b(P) \leq p^{6.5a}$ and $\text{dl}(P) \leq \log_2 a + 5 + \log_2 6.5$.*

Proof. Let $T = O_\infty(G)$, the maximal normal subgroup of G . Since $\mathbf{F}(G) \subseteq T$, $\mathbf{F}(T) = \mathbf{F}(G)$. Since $T \triangleleft G$, p^{a+1} does not divide $\lambda(1)$ for all $\lambda \in \text{Irr}(T)$. Thus by [23, Remark of Corollary 5.3], $|T : \mathbf{F}(G)|_p \leq p^{2.5a}$.

Let $\tilde{G} = G/O_\infty(G)$ and $\bar{G} = \tilde{G}/F^*(\tilde{G})$. It is clear that $F^*(\tilde{G})$ is a direct product of finite non-abelian simple groups. By Proposition 2.4(1), there exists $N \triangleleft \tilde{G}$ where $N \subseteq \mathbf{F}_2(\tilde{G})$ such that for any $P \in \text{Syl}_p(\tilde{G})$, we have $\mathbf{C}_P(v) \subseteq N$, and the Sylow p -subgroup of $N\mathbf{F}(\tilde{G})/\mathbf{F}(\tilde{G})$ is abelian. It is clear that we may find $\bar{\gamma} \in \text{Irr}(\bar{G})$ such that $|\bar{G} : N|_p$ divides $\bar{\gamma}(1)$.

Let $P/\mathbf{F}(\tilde{G})$ be a Sylow p -subgroup of $N\mathbf{F}(\tilde{G})/\mathbf{F}(\tilde{G})$. Where $Y = O_{p'}(\mathbf{F}(\tilde{G}))$, observe that $W = \text{Irr}(Y/\Phi(Y))$ is a faithful and completely reducible $P/\mathbf{F}(\tilde{G})$ -module. By Gow's regular orbit theorem [7, 2.6], we have $\mu \in W$ such that $\mathbf{C}_P(\mu) = \mathbf{F}(\tilde{G})$. We may view μ as a character of the preimage X of Y in \tilde{G} . Take $\bar{\alpha} \in \text{Irr}(\bar{G})$ lying over μ , and $\bar{\alpha}$ lies over an irreducible character $\bar{\psi}$ of P lying over μ . Clearly, $\bar{\psi}(1)_p \geq |N\mathbf{F}(\tilde{G})/\mathbf{F}(\tilde{G})|_p$. As P is normal in G , we have $\bar{\alpha}(1)_p \geq \bar{\psi}(1)_p \geq |N\mathbf{F}(\tilde{G})/\mathbf{F}(\tilde{G})|_p$.

Let P_1 be the Sylow p -subgroup of $\mathbf{F}(\tilde{G}) \cap N$. By Lemma 2.1, we may find $\nu \in \text{Irr}(F^*(\tilde{G}))$ such that $\mathbf{C}_{P_1}(\nu) = 1$. Thus by a similar argument as before, we may find an irreducible character $\bar{\beta}$ of \bar{G} such that $\bar{\beta}(1)_p \geq |N \cap \mathbf{F}(\tilde{G})|_p$.

Thus $|G : \mathbf{F}(G)|_p \leq p^{5.5a}$. If $P \in \text{Syl}_p(G)$, then $b(P) \leq |P : O_p(G)||b(O_p(G))| = |G : \mathbf{F}(G)|_p |b(O_p(G))| \leq p^{5.5a} p^a = p^{6.5a}$.

Now, we want to prove the last part of the statement. By [13, Theorem 12.26] and the nilpotency of P , we have that P has an abelian subgroup B of index at most $b(P)^4$. By [20, Theorem 5.1], we deduce that P has a normal abelian subgroup A of index at most $|P : B|^2$. Thus, $|P : A| \leq |P : B|^2 \leq b(P)^{8s}$, where $b(P) = p^s$. By [11, Satz III.2.12], $\text{dl}(P/A) \leq 1 + \log_2(8s)$ and so $\text{dl}(P) \leq 2 + \log_2(8s) = 5 + \log_2(s)$. Since s is at most $6.5a$, the result follows. \square

We now state the conjugacy analogs of Theorem 3.1. Given a group G , we write $b^*(G)$ to denote the largest size of the conjugacy classes of G .

Theorem 3.2. *Let G be a p -solvable group where p is a prime and $p \geq 5$. Suppose that p^{a+1} does not divide $|C|$ for all $C \in \text{cl}(G)$ and let $P \in \text{Syl}_p(G)$, then $|G : \mathbf{F}(G)|_p \leq p^{5.5a}$, $b^*(P) \leq p^{6.5a}$ and $|P'| \leq p^{6.5a(6.5a+1)/2}$.*

Proof. The proof of the first statement goes similarly as the previous one. Write $N = O_p(G)$. It is clear that $|N : \mathbf{C}_N(x)|$ divides $|G : \mathbf{C}_G(x)|$ for all $x \in G$. Thus, if we take $x \in P$ we have that

$$|\text{cl}_P(x)| = |P : \mathbf{C}_P(x)| \leq |P : N||N : \mathbf{C}_N(x)| \leq p^{5.5a} p^a = p^{6.5a}$$

Finally, to obtain the bounds for the order of P' it suffices to apply a theorem of Vaughan-Lee [12, Theorem VIII.9.12]. \square

The following is a corrected version of [22, Theorem 5.1] (Note the $t \leq 19$ in the original statement should be $t \leq 15$).

Theorem 3.3. *If G is solvable, there exists a product $\theta = \chi_1 \dots \chi_t$ of distinct irreducible characters χ_i of G such that $|G : \mathbf{F}(G)|$ divides $\theta(1)$ and $t \leq 15$. Furthermore, if $|\mathbf{F}_8(G)|$ is odd then we can take $t \leq 3$ and if $|G|$ is odd we can take $t \leq 2$.*

For the case $p = 3$, we have the following results. The proof is similar as before but using Theorem 3.3 instead of [23, Remark of Corollary 5.3], and Proposition 2.4(2) instead of Proposition 2.4(1).

Theorem 3.4. *Let G be a p -solvable group where p is a prime and $p = 3$. Suppose that p^{a+1} does not divide $\chi(1)$ for all $\chi \in \text{Irr}(G)$ and let $P \in \text{Syl}_p(G)$, then $|G : \mathbf{F}(G)|_p \leq p^{20a}$, $b(P) \leq p^{21a}$ and $\text{dl}(P) \leq \log_2 a + 5 + \log_2 21$.*

Theorem 3.5. *Let G be a p -solvable group where p is a prime and $p = 3$. Suppose that p^{a+1} does not divide $|C|$ for all $C \in \text{cl}(G)$ and let $P \in \text{Syl}_p(G)$, then $|G : \mathbf{F}(G)|_p \leq p^{20a}$, $b^*(P) \leq p^{21a}$ and $|P'| \leq p^{21a(21a+1)/2}$.*

4. DISCUSSIONS

In order to improve the bounds of the results in this paper, one might need to study the situation when a p -solvable group acts on a field of characteristic does not equal to p , and hope that a similar result as [23, Theorem 3.3] holds. Also, a strengthened version of Lemma 2.1 would also be helpful in improving the bounds.

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